7 - Sample space, events, counting

Statistics (experiments, surveys) is **empirical**.
Probability theory models the **hidden** process that generates this observed outcome.

Outcomes: All possible results of an experiment. Here are some observed outcomes:	**Sample space**: Set of all outcomes.	**Event**: A subset of the sample space that interests us.
Experiment 1: Flip a coin H, T, H, H, ...	{H, T}	All events: {}, {H}, {T}, {H, T}
Experiment 2: Flip 2 coins HT, TH, HH, HT, ...	{H, T} x {H, T}	{Two heads} = {(H, H)} {One head} = {(H, T), (T, H)} {≥ 1 tail} = {(T, H), (H, T), (T, T)}
Experiment 3: Draw a card Face, No face, No face, No face, No face, No face, Face, No face, No face, ... Faces come up about 23% of the time	To get face vs no-face cards, you'd think I was drawing from a deck of 52 playing cards: {A, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K} x {♠, ♥, ♦, ♣}	{Face cards} = {J, Q, K} x {♠, ♥, ♦, ♣} {Heart cards} = {A, 2, ..., J, Q, K} x {♥}

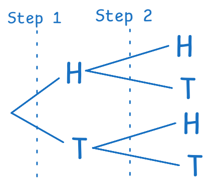
Experiment 3 was actually generated by sampling only 13 cards of the same suit (not the whole deck of 52 cards).

Moral: different probability models (with different sample spaces) can produce the same observed outcomes.

How to count occurrences of observed outcomes?

> **Step Rule of Counting.** (aka Multiplication Rule)
> If an experiment has two steps (step 1: m outcomes; step 2: n outcomes).
> Then the experiment has $m \cdot n$ outcomes.

Example 1. How many outcomes are possible by flipping two coins?
(Aka: What is the size of this sample space?)



Answer: Step 1: Flip 1st coin (2 outcomes).
Step 2: Flip 2nd coin (2 outcomes).
Overall: 2 x 2 = 4 outcomes.

Step Rule works the same way for more than 2 steps:

Example 2. A box has balls labelled A, B, C. How many outcomes are possible by sample three balls "with replacement"? (i.e. put the ball back after sampling)

Answer: Step 1: Choose one of A, B, C. (3 options)
Step 2: Same. (3 options)
Step 3: Same. (3 options) ... overall 3 x 3 x 3 = 27 options.

Example 3. What if now we sample three balls "without replacement"?

Answer: Step 1: Choose one of the balls A, B, C. (3 options)
Step 2: Choose one of remaining 2 balls (2 options)
Step 3: (1 options) ... overall 3 x 2 x 1 = 6 options.

Definition. The **factorial** of n is $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$

So: $2! = 2 \times 1 = 2$ and $3! = 3 \times 2 \times 1 = 6$

We can reinterpret Example 3 as counting rearrangements or permutations of distinct letters:

Example 4. How many rearrangements of the word "RANDOM" are there? Answer: 6!

To count permutations of indistinct letters, divide by the ways identical letters could be permuted:

Example 5. How many rearrangements of the word "MOMMA" are there?

$M_1OM_2M_3A$	$M_1OM_3M_2A$	$M_2OM_1M_3A$	$M_2OM_3M_1A$	$M_3OM_1M_2A$	$M_3OM_2M_1A$	$\rightarrow MOMMA$
$M_1M_2M_3AO$	$M_1M_3M_2AO$	$M_2M_1M_3AO$	$M_2M_3M_1AO$	$M_3M_1M_2AO$	$M_3M_2M_1AO$	$\rightarrow MMMAO$
$M_1M_2M_3OA$	$M_1M_3M_2OA$	$M_2M_1M_3OA$	$M_2M_3M_1OA$	$M_3M_1M_2OA$	$M_3M_2M_1OA$	$\rightarrow MMMOA$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	

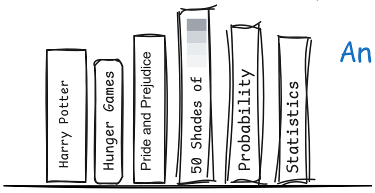
Answer: If M's are distinct, there are 5! such words. But the 3 M's are considered identical, so we group their 3! rearrangements together.
Answer: 5!/3!

Example 6. How many rearrangements of the word "STATISTICS" are there?

Answer: Of the 10 letter, there are 3 S's, 3 T's, 2 I's, so $\frac{10!}{3!3!2!}$

Count the unordered selection of k things from n things via "Yes"/"No" assignment:

Example 7. How many ways can you select 3 books from 6 books to bring with you?



Answer: Selecting 3 books from 6 is encoded as 3 Y's and 3 N's
e.g. selecting first 3 books is encoded as YYYNNN.
Number of such selections = permutations of YYYNNN
= 6!/(3!3!)

Definition. **k-combination** counts the unordered selection of k things from n distinct things, sampled without replacement: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ ← "binomial coefficient" or "n choose k"

Example 8. How many ways can you select 2 villagers from a district of 8,000 in the Hunger Games?

Answer: This is 8000 choose 2.

Example 9. Flip 7 coins. How many outcomes have exactly 3 heads?

Answer: This is 7 choose 3, or # of permutations of HHHTTTT.